

# Partitions and their lattices

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## Abstract

Ferrers graphs and tables of partitions are treated as vectors. Matrix operations are used for simple proofs of identities concerning partitions. Interpreting partitions as vectors gives a possibility to generalize partitions on negative numbers. Partitions are then tabulated into lattices and some properties of these lattices are studied. There appears a new identity counting isoscele Ferrers graphs. The lattices form the base for tabulating combinatorial identities.

## 1 Preliminary Notes

Partitions of a natural number  $m$  into  $n$  parts were introduced into mathematics by Euler. The analytical formula for finding the number of partitions was derived by Ramanudjan and Hardy [1]. Ramanudjan was a mathematical genius from India. He was sure that it was possible to calculate the number of partitions exactly for any number  $m$ . He found the solution in cooperation with his tutor, the English mathematician Hardy. It is rather complicated formula derived by higher mathematical techniques. We will use only simple recursive methods for different relations between partitions.

Steve Weinberg in his lecture [2] about importance of mathematics for physics mentioned that partitions got importance for theoretical physics, even if Hardy did not want to study practical problems. But partitions were used in physics before Hardy by Boltzmann [3]. He used this notion for splitting  $m$  quanta of energy between  $n$  particles in connection with his notion of entropy. He called partitions complexions, considering them to be orbits in phase space. His idea was forgotten.

A partition splits a number  $m$  into  $n$  parts which sum is equal to the number  $m$ , say  $7 : 3, 2, 1, 1$ . A partition is an ordered set. Its objects, *parts*, are written in a row in decreasing order:

$$m_{j-1} \geq m_j \geq m_{j+1} .$$

If we close a string of parts into brackets, we get a  $n$  dimensional vector row  $p = (3, 2, 1, 1)$ . From a partition vector, another vectors having equivalent structure of elements, for example.  $r = (1, 2, 1, 3)$ , are obtained by permuting, simple changing of ordering of vector elements. The partitions are thus indispensable for obtaining combinatorial identities, for ordering points of plane simplexes having constant sums of its constituting vectors.

All unit permutations of a vector have the same length. Therefore different partitions form bases for other vectors composed from the same parts. Vectors belonging to the same partition of  $p$  into three parts are connected with other points of the three dimensional simplex by circles. In higher dimensions the circles become spheres and therefore we will call an ordered partition the *partition orbit* or simply orbit.

The number of vectors in partitions will be given as  $n$ , the size of the first vector as  $m_1$ . The bracket  $(m, n)$  means all partitions of the number  $m$  into at most  $n$  parts. Because we write a partition as a  $n$  dimensional vector we allow zero parts in a partition to fill empty places of the vector. It is a certain innovation against the tradition which will be very useful. But it is necessary to distinguish strictly both kinds of partitions, with zeroes and without them.

## 2 Transposing and Transversing

Transposition of matrix vectors is the basic operation on matrices. It changes simply the row indices  $i$  and column indices  $j$  of all matrix elements

$$\mathbf{M}^T \rightarrow m_{ij}^T = m_{ji} \quad (1)$$

Figure 1: Transposing (A) and transversing (B) of matrices

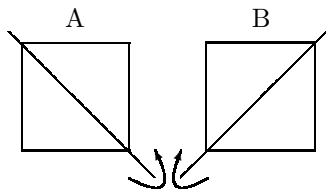
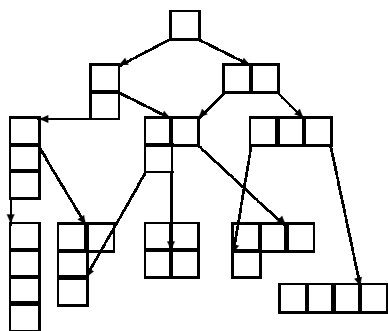


Figure 2: Ferrers graphs construction. New boxes are added to free places

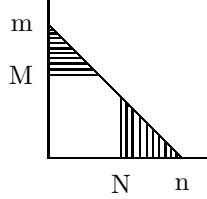


The second operation introduced here, *transversing*, is not used in textbooks but we need it to prove simply, without calculations, some combinatorial identities concerning partitions. The transversing changes the ordering of both indices, that means rows and columns are counted backwards. If transposing rotates matrix elements around the main diagonal  $m_{11} \rightarrow m_{nn}$ , transversing rotates them around the diagonal (its name will be transversal)  $m_{1n} \rightarrow m_{n1}$  (Fig. 1). We look on the matrix's most distant corner as its starting point.

### 3 Ferrers Graphs

Ferrers graphs are used in the theory of partitions for many proofs based simply on their properties. Ferrers graphs are tables (see Fig. 2) containing  $m$  objects,

Figure 3: Truncation of partitions by restrictions of rows and columns



each object in its own box. The square boxes are arranged into columns in nonincreasing order  $m_j \geq m_{j+1}$  with the sum

$$\sum_{j=1}^n m_j = \sum_{k=0}^{\infty} n_k m_k = m . \quad (2)$$

If partitions contain equal parts, it is possible to count them together using the index  $k$  and their number  $n_k$ .

It is obvious that a Ferrers graph, completed to an quadrangle with zero positions, is a matrix  $\mathbf{F}$  which has its unit elements arranged consecutively in the initial rows and columns.

Introducing Ferrers graphs as matrices, we come necessarily to the notion of *restricted partitions*. The parts of a partitions can not be greater than the number of rows of the matrix and the number of parts greater than the number of its columns.

The interpretation of restrictions is geometrical. The part  $m_{max}$  determines the side of a cube,  $n$  is its dimension, see Fig. 3.

A sophistication of the notation distinguishes the partitioned number  $M$  and the number of rows  $m$  in the matrix  $\mathbf{F}$ . The unrestricted number of partitions  $p(M)$  is equal to the number of restricted partitions when restricting conditions are loose then  $m \geq M$  and  $n \geq M$ :

$$p(M)_{unrestricted} = p(M, M, M) . \quad (3)$$

We write here first the number of rows  $m$ , then the number of parts  $n$ , here considered as equal to  $m$  and at last the sum of unit elements (the number of filled boxes)  $M$ .

An important property of restricted partitions is determined by transposing Ferrers graphs  $\mathbf{F} \rightarrow \mathbf{F}^T$ :

$$p(m, n, M) = p(n, m, M) . \quad (4)$$

The partitions are conjugated. The number of partitions into exactly  $n$  parts with the greatest part  $m$  is the same as the number of partitions into  $m$  parts having the greatest part  $n$ . This is simple transposing of  $\mathbf{F}$ .

A Ferrers graph can be subtracted from the matrix containing only unit elements (defined as  $\mathbf{J}\mathbf{J}^T$ ,  $\mathbf{J}$  being the unit column), and the resulting matrix transversed (Tr), for example

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{\text{Tr}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The relation between the number of restricted partitions of two different numbers is obtained according to the following equation

$$p(m, n, M) = p(n, m, mn - M) . \quad (5)$$

This identity was derived by an operation very useful for acquiring elements of partition schemes (see later) and restricted partitions of all kinds. A restricted partition into exactly  $n$  parts, having  $m$  as the greatest part, has  $(m+n-1)$  units bounded by elements forming the first row and column of the corresponding Ferrers graph (Fig. 2). Only  $(M - m - n + 1)$  elements are free for partitions in the restricted frame  $(m - 1)$  and  $(n - 1)$ . Therefore

$$p(m, n, M) = p(m - 1, n - 1, M - m - n + 1) . \quad (6)$$

For example:  $p(4, 3, 8) = p(3, 2, 2) = 2$ . The corresponding partitions are 4,3,1 and 4,2,2; or 2,0 and 1,1; respectively. This formula can be used for finding all restricted partitions.

It is rather easy when the difference  $(M - m - n + 1)$  is smaller than the restricting values  $m$  and  $n$  or at least one from the restricting values. The row and column sums of partially restricted partitions having the other constrain constant (shown as an asterics), where either  $n$  or  $m$  can be 1 till  $M$  are:

$$p(m, *, M) = \sum_{j=1}^M p(m, j, M) \quad (7)$$

$$p(*, n, M) = \sum_{i=1}^M p(i, n, M) . \quad (8)$$

Before we examine restricted partitions in more detail, tables of unrestricted and partially restricted partitions will be introduced.

## 4 Partition Matrices

Partially restricted partitions can be obtained from unrestricted partitions by subtracting a row of  $n$  units or a column of  $m$  units. This gives us the recursive formula for the number of partitions as a sum of two partitions

$$p(*, N, M) = p(*, N - 1, M - 1) + p(*, N, M - N - 1) . \quad (9)$$

All partitions into exactly  $N$  parts are divided into two sets. In one set are partitions having in the last column 1, their number is counted by the term  $p(*, N - 1, M - 1)$  which is the number of partitions of the number  $(M - 1)$  into exactly  $(N - 1)$  parts to which 1 was added on the  $n$  th place and in other set are partitions which have in the last column 2 and more. They were obtained

by adding the unit row  $\mathbf{J}^T$  with  $n$  unit elements to the partitions of  $(M - N)$  into  $N$  parts. Their number can be found in the same column column  $n$  places above.

A similar formula can be deduced for partitions of  $M$  into at most  $N$  parts. These partitions can have zero at least in the last column or they are partitioned into  $n$  parts exactly:

$$p(*, * = N, M) = p(*, * = N - 1, M) + p(*, * = N, M - N) . \quad (10)$$

The term  $p(*, * = N - 1, M)$  are partitions of  $M$  into  $(N - 1)$  parts transformed in partitions into  $N$  parts by adding zero in the  $n$ -th column, the term  $p(*, * = N, M - N)$  are partitions of  $(M - 1)$  into  $N$  parts to which the unit row was added.

To formulate both recursive formulas more precisely, we had to define an apparently paradoxical partition at first:

$$p(0, 0, 0) = 1 .$$

What it means? A partition of zero into zero number of parts. This partition represents the empty space of dimension zero. This partition is justified by its limit. We write  $n = 0^0$  and find the limit:

$$\lim 0^0 = \lim_{x \rightarrow \infty} (1/x)^0 = 1/x^0 = 1 . \quad (11)$$

We get two following tables of partitions

Table 2 is obtained from the Table 1 as partial sums of its rows, it means, by multiplying with the unit triangular matrix  $\mathbf{T}^T$  from the right. The elements of the matrix  $\mathbf{T}^T$  are

$$h_{ij} = 1 \text{ if } j \geq i \quad h_{ij} = 0 \text{ if } j < i . \quad (12)$$

On the other hand, the Table 2 is obtained from the Table 1 by multiplying with a matrix  $\mathbf{T}^{-T}$  from the right. The inverse elements are

Table 1: Partitions into exactly  $n$  parts

n	0	1	2	3	4	5	6	$\Sigma$
m=0	1							1
1		1						1
2		1	1					2
3		1	1	1				3
4		1	2	1	1			5
5		1	2	2	1	1		7
6		1	3	3	2	1	1	11

Table 2: Partitions into at most  $n$  parts

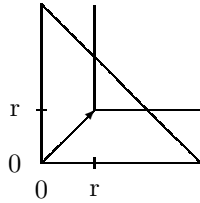
n	0	1	2	3	4	5	6
m=0	1	1	1	1	1	1	1
1		1	1	1	1	1	1
2		1	2	2	2	2	2
3		1	2	3	3	3	3
4		1	3	4	5	5	5
5		1	3	5	6	7	7
6		1	4	7	9	10	11

$$h_{ii}^{-1} = 1, \quad h_{i,i+1}^{-1} = -1, \quad h_{ij} = 0, \text{ otherwise.} \quad (13)$$

Notice, that the elements of the Table 2 right of the diagonal remain constant. They are equal to the row sums of the Table 1. Increasing the number of zeroes does not change the number of partitions.

When we multiply Table 1 by the matrix  $\mathbf{T}^{-T}$  again, we obtain partitions

Figure 4: Limiting of partition orbits. The lowest allowed part  $r$  shifts the plane simplex





having as the smallest allowed part the number 2. The effect of these operators can be visualized on the 2 dimensional complex, the operators shift the border of counted orbits (Fig. 4). The operator  $\mathbf{T}^T$  differentiates  $n$  dimensional complexes, shifting their border to positive numbers and cutting lower numbers. Zero forms the natural base border.

## 5 Partitions with Negative Parts

Operations with tables of partitions lead to a thought, what would happen with partitions outside the positive cone of nonnegative numbers. Thus let us allow the existence of negative numbers in partitions, too<sup>1</sup>.

If the number of equal parts  $n_k$  is written as the vector row under the vector formed by the number scale, the number of partitions is independent on shifts of the number scale, see Table 3. Partitions, shown in the bottom part of the table, are always derived by shifting two vectors, one 1 position up, the other 1 position down. Each partition corresponds to a vector. If we write them as columns then their scalar product with the number scale, forming the vector row  $\mathbf{m}^T$ , gives constant sum:

$$\mathbf{m}^T \mathbf{p} = \sum_{k \geq r} m_k n_k = m . \quad (14)$$

There is an inconsistency in notation, elements of the vector  $\mathbf{p}$  are numbers of vectors having the same length and the letter  $n$  with an index  $k$  is used for them. For values of the number scale the letter  $m$  is used with the common index  $k$  which goes from the lowest allowed value of parts  $r$  till the highest possible value. The index  $k$  runs to infinity but all too high values  $n_k$  are zeroes.

Using different partition vectors and different vectors  $\mathbf{m}$  we get the following examples:

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<sup>1</sup>The negative parts can be compared in physics with antiparticles. Since an annihilation liberates energy, it does not annihilate it, the energy of the Universe is infinite. Speculations about existence of antiworlds, formed only by antiparticles balancing our world, can be formulated as doubt if the Universe is based in the natural cone of the space.

Table 3: Partitions as vectors

Parameter	r						
Vector <b>m</b>	-2	-1	0	1	2	3	<b>mp</b> =
	-1	0	1	2	3	4	
	0	1	2	3	4	5	
	1	2	3	4	5	6	
	2	3	4	5	6	7	
Vector p	4					1	
	3	1			1		
	3		1	1			
	2	2		1			
	2	1	2				
	1	3	1				
	1	2	2				
		5					

$$(4 \times -2) + (1 \times 3) = -5$$

$$(3 \times -1) + (1 \times 0) + (1 \times 3) = 0$$

$$(3 \times 0) + (1 \times 2) + (1 \times 3) = 5$$

$$(2 \times 1) + (1 \times 2) + (2 \times 3) = 10$$

$$(1 \times 2) + (3 \times 3) + (1 \times 4) = 15.$$

The parameter  $r$  shifts the table of partitions, its front rotates around the zero point. If  $r$  were  $-\infty$ , then  $p(-\infty, 1) = 1$  but  $p(-\infty, 2)$  were undetermined, because a sum of a finite number with an infinite number is again infinite. The parameter  $r$  will be written to a partition as its upper index to show that different bases of partitions are differentiating plane simplexes.

## 6 Partitions with Inner Restrictions

Partitions were classified according to the minimal and maximal allowed values of parts, but there can be restrictions inside the number scale, it can be prescribed that some values are forbidden. It is easy to see what this means:

Table 4: Odd, even, and mixed partitions

n	Number of odd partitions									Sums			
	1	2	3	4	5	6	7	8	9	Odd	Even	Mixed	p(m)
m=1	1									1	0	0	1
2		1								1	1	0	2
3	1		1							2	0	1	3
4		1		1						2	2	1	5
5	1		1		1					3	0	4	7
6		2		1		1				4	3	4	11
7	1		2		1		1			5	0	10	15
8		2		2		1		1		6	5	11	22
9	1		3		2		1		1	8	0	22	30

The plane simplex has holes, some orbits cannot be realized and its  $(n - 1)$  dimensional body is thinner than the normal one.

It is easy to find the number of partitions in which all parts are even. It is not possible to form an even partition from an uneven number, therefore:

$$p_{\text{even}}(2n) = p_{\text{unrestricted}}(n) . \quad (15)$$

A more difficult task is finding the number of partitions in which all parts are odd. The rejected partitions contain mixed odd and even parts. The relation between different partitions is etermined as

$$p_{\text{unrestricted}}(n) = p_{\text{odd}}(n) + p_{\text{even}}(n) + p_{\text{mixed}}(n) . \quad (16)$$

The corresponding lists are given in Table 4

Notice how the scarce matrix of odd partitions is made from Table 1. Its elements, except the first one in each column, are shifted down on cross diagonals. An odd number must be partitioned into an odd number of odd parts and an even number into even number of odd parts. Therefore the matrix can be filled only in half. The recurrence is given by two possibilities how to increase the number  $m$ . Either we add odd 1 to odd partitions of  $(m - 1)$  with exactly  $(j - 1)$  parts or we add  $2j$  to odd numbers of partitions of  $(m - 2j)$  with exactly

Table 5: Partitions with unequal parts

n	1	2	3	4	$\Sigma$	Difference ( $n_{odd} - n_{even}$ )
m=1	1				1	1
2	1				1	1
3	1	1			2	0
4	1	1			2	0
5	1	2			3	-1
6	1	2	1		4	0
7	1	3	1		5	-1
8	1	3	2		6	0
9	1	4	3		8	0
10	1	4	4	1	10	0
11	1	5	5	1	12	0
12	1	5	7	2	15	1

$j$  parts. The relation is expressed as

$$o(i, j) = p[(i + j)/2, j] . \quad (17)$$

Partitions with all parts unequal are important, because their transposed Ferrers graphs have the greatest part odd, when the number of parts is odd, and even, when the number of parts is even. For example

$$\begin{array}{rcl}
 10 & & \\
 & 9,1 & \\
 & 8,2 & \\
 & 7,3 & 7,2,1 \\
 & 6,3 & 6,3,1 \\
 & & 5,4,1 \\
 & & 5,3,2 \\
 & & 4,3,2,1
 \end{array}$$

The partitions with unequal parts can be tabulated as in Table 5. Notice that the difference of the even and odd columns partitions is mostly zeroes and only sometimes  $\pm 1$ . The importance of this phenomenon will be explained later. The number of partitions with unequal parts coincide with the partitions which all parts are odd.

The differences are due to Franklin blocks with growing minimal parts and

Table 6: Partitions Differentiated According to Unit Parts

n	0	1	2	3	4	5	6
m=0	1						
1	0	1					
2	1	0	1				
3	1	1	0	1			
4	2	1	1	0	1		
5	2	2	1	1	0	1	
6	4	2	2	1	1	0	1

growing number of parts (their transposed notation is used), which are minimal in that sense that their parts differ by one, the shape of corresponding Ferrers graphs is trapeze:

$$\begin{array}{ccc}
 (1) & (11) & 1, 2 \\
 \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right) & \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array} \right) & 5, 7 \\
 \left( \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array} \right) & \left( \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{array} \right) & 12, 15
 \end{array}$$

## 7 Differences According to Unit parts

We have arranged restricted partitions according to the number of nonzero parts in Table 1. It is possible to classify partitions according the number of vectors in the partition having any value. Using value 1, we get another kind of partition differences as in Table 6 .

The elements of the table are:

$$p_{i0} = p(i) - p(i-1), \quad p_{ij} = p_{i-1,j-1}, \text{ otherwise .} \quad (18)$$

Table 6 is obtained from the following Table 7 of rows of unrestricted partitions by multiplying it with the matrix  $\mathbf{T}^{-1}$ . The zero column of the Table 6 is the difference of two consecutive unrestricted partitions according to  $m$ . To

Table 7: Partitions and their Euler inversion

	Partition table						Euler inversion					
j	0	1	2	3	4	5	0	1	2	3	4	5
i=0	1						1					
1	1	1					-1	1				
2	2	1	1				-1	-1	1			
3	3	2	1	1			0	-1	-1	1		
4	5	3	2	1	1		0	0	-1	-1	1	
5	7	5	3	2	1	1	1	0	0	-1	-1	1

all partitions of  $p(m - k)$  were added  $k$  ones. The partitions in the zero column contain only numbers greater than 1. These partitions can not be formed from lower partitions by adding ones and they are thus a difference of the partition function according to the number  $n_1$ . Since Table 6 is composed, it is the product of two matrices, its inverse is composed, too.

## 8 Euler Inverse of Partitions

If we write successive partitions as column or row vectors as in Table 7, which elements are

$$p_{ij} = p(i - j + 1) , \quad (19)$$

we find rather easily its inverse matrix which is given in the second part of the same Table.

The nonzero elements in the first column of the Euler inversion (and similarly in the next columns which are only shifted down one row) appear at indices, which can be expressed by the Euler identity concerning the coefficients of expansion of

$$(1 - t)(1 - t^2)(1 - t^3)\dots = 1 + \sum_{i=1}^{\infty} (-1)^i [t^{3i^2-i}/2 + t^{3i^2+i}/2] . \quad (20)$$

For example the last row of the partition Table 7 is eliminated by multiplying it with the Euler inversion as:

$$(7 \times 1) + (5 \times -1) + (3 \times -1) + (2 \times 0) + (1 \times 0) + (1 \times 1) = 0$$

when  $i = 1$ , there is the pair of indexes at  $t^1, t^2$  with negative sign; for  $i = 2$  the pair is  $t^5, t^7$ ; for  $i = 3$  the pair is  $t^{-12}, t^{-15}$  and so on. These numbers are the distances from the base partition. The inverse matrix becomes scarcer as  $p(m)$  increases, as it was already shown in Franklin partitions above. All inverse elements are  $-1, 0, 1$ . The nonzero elements of the Euler polynomial are obtained as sums of the product

$$\prod_{i=1}^{\infty} (1 - t^i) . \quad (21)$$

This is verified by multiplying several terms of the infinite product. If we multiply the Euler polynomial with its inverse function

$$\prod_i = 1^{\infty} (1 - t^i)^{-1} , \quad (22)$$

we obtain 1. From this relation follows that partitions are generated by the inverse Euler function which is the *generating function* of partitions. Terms  $t^i$  must be considered as representing unequal parts.

The Euler function has all parts  $t^i$  different. We have constructed such partitions in Table 5. If the coefficient at  $t^i$  is obtained as the product of the even number of  $(1 - t^i)$  terms then the sign is positive, and if it is the result of the uneven number of terms then the sign is negative. The coefficients are determined by the difference of the number of partitions with odd and even number of unequal parts. This difference can be further explained according to Franklin using Ferrers graphs.

All parts in  $p(n)$  having as at least one part equal to 1 are obtained from  $p(n-1)$ . The difference  $p(n) - p(n-1)$  is due to some terms of  $p(n-2)$ . We must add to each partition of  $p(n-2)$  2, except all partitions of  $p(n-2)$  containing 1. These must be either removed or used in transposed form using transposed Ferrers graphs, since big parts are needed. One from the pair of conjugate

Table 8: Inverse matrix to partitions into n parts

n	1	2	3	4	5	6
m=1	1					
2	-1	1				
3	0	-1	1			
4	1	-1	-1	1		
5	0	1	-1	-1	1	
6	0	1	0	-1	-1	1

partitions is superfluous. These unused partitions must be subtracted. For example for  $p(8)$ :

$$\begin{array}{ll}
 6; & 1^6; \\
 \underline{51}; & 21^4; \\
 42; & 2^2 1^2; \\
 \underline{33}; & 2^3; \\
 41^2; & \underline{31^3}; \\
 \underline{321}; & 
 \end{array}
 \quad
 \text{Formed :}
 \quad
 \begin{array}{ll}
 8; & 62; \\
 53; & \\
 44; & 2^4; \\
 3^2 2; & \\
 42^2; & 
 \end{array}$$

Leftovers (underlined above):

$$p(1) + 5: 51; \quad p(3) + 3: 33; 321; 31$$

are obtained by subtracting the largest part from corresponding partition. Two must be added to the subtracted part. We get  $p(8-5)$  and  $p(8-7)$  as the corrections.

## 9 Other Inverse Functions of Partitions

We already met other tables of partitions which have inverses because they are in lower triangular form. The inverse to the Table 1 is Table 8.

The inverse to Table 6 is Table 9.

Whereas the columns of the Table 8 are irregular and elements of each column must be found separately, columns of the Table 9 repeat as they are only shifted in each column one row down, similarly as the elements of their parent matrix are. They can be easily found by multiplying the matrix of the Euler function (Table 7) by the matrix  $\mathbf{T}$  from the left.



Table 9: Inverse matrix of unit differences

n	1	2	3	4	5	6
m=1	1					
2	0	1				
3	-1	0	1			
4	-1	-1	0	1		
5	-1	-1	-1	0	1	
6	0	-1	-1	-1	0	1

Table 10: Orbits in 3 dimensional cubes

Edge size	0	1	2	3
m=0	000	000	000	000
1		100	100	100
2		110	200; 110	210; 110
3		111	210; 111	300; 210; 111
4			220; 211	310; 220; 211
5			221	320; 311; 221
6			222	330; 321; 222
7				331; 322
8				332
9				333

## 10 Partition Orbits in m Dimensional Cubes

Restricted partitions have a geometric interpretation: They are orbits of  $n$  dimensional plane complices truncated into cubes with the sides  $(m - 1)$  as on Fig. 3.

We can count orbits even in cubes. It is a tedious task if some special techniques are not applied, since their number depends on the size of the cube. For example for the 3 dimensional space we get orbits as in Table 10 .

The Equation 3 can be applied for cubes. It shows their important property, they are symmetrical along the main diagonal, going from the center of the coordinates, the simplex  $n^0$  to the most distant vertex of the cube in which all  $n$  coordinates are  $(m - 1)$ . The diagonal of the cube is represented on Table 10 by  $k$  indices. Moreover, a cube is convex, therefore

$$M \leq mn/2 \text{ then } p(m, n, M) \geq p(m, n, M - 1) \quad (23)$$

and if

$$M \geq mn/2 \text{ then } p(m, n, M) \leq p(m, n, M - 1) . \quad (24)$$

Here we see the importance of restricted partitions. From Table 10, we find the recurrence, which is given by the fact that in a greater cube the lesser cube is always present as its base. New orbits which are on its enlarged sides are added to it. But it is enough to know orbits of one enlarged side, because the other sides are formed by these orbits. The enlarged side of a  $n$  dimensional cube is a  $(n - 1)$  dimensional cube. The recurrence relation for partitions in cubes is thus

$$p(m, n, M) = p(m - 1, n, M) + p(m, n - 1, M) . \quad (25)$$

This recurrence will be explained later more thoroughly.

## 11 Generating Functions of Partitions in Cubes

The generating function of partitions is simply the generating function of the infinite cube in the Hilbert space, which sides have different meshes:

$$\text{Parts 1 : } (1 + t_1^1 + t_1^2 + \dots t_1^\infty) \quad (26)$$

$$\text{Parts 2 : } (1 + t_2^1 + t_2^2 + \dots t_2^\infty) \quad (27)$$

and so on till

$$\text{Parts } \infty : (1 + \dots t_\infty^1) . \quad (28)$$

When the multiplication's for all parts are made and terms on consecutive plane simplexes counted, we get:

$$1 + t_1^1 + [t_2^1 + t_1^2] + [t_3^1 + \dots] \quad (29)$$

The generating function of restricted partitions is obtained by canceling unwanted (restricted) parts. Sometimes the generating function is formulated in an inverse form. The infinite power series are replaced by the differences  $(1 - t_k^{-1})$ . This is possible if we consider  $t$  to be only a dummy variable. For example, the generating function of the partitions with unequal unrepeated parts is given by the product

$$u(t) = \prod_{k=1}^{\infty} (1 - t_k) \quad (30)$$

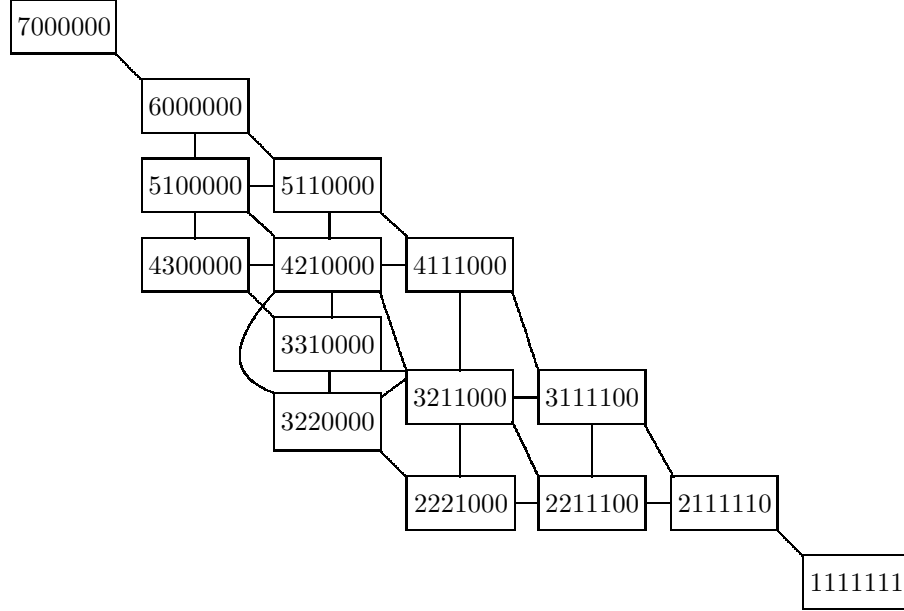
The mesh of the partition space is regular, it covers all numbers. The number of partitions is obtained by recursive techniques. But it is a very complicated function, if it is expressed in one closed formula, as the Ramanudjan-Hardy function is. The partitions form a carcass of the space. We will be interested, how the mesh of partitions is filled into the space which all axes have unit mesh and which contains also vector strings.

## 12 Partition Schemes

Multidimensional plane simplexes are complicated objects and it is necessary to find tools how to analyze them. To draw them is impossible, as it was mentioned, because their parts are layered in our 3 dimensional world over themselves.

We already classified orbits in plane simplexes according to the number  $k$  of nonzero parts. This number shows the dimensionality of subsimplices, their vertices, edges, and  $(k-1)$  dimensional bodies. Lately we introduced the number of unit vectors as a tool differentiating the simplex. Now we arrange partitions as two dimensional tables. These tables will be called *partition schemes*.

Figure 5: Lattice of partition orbits (7,7)



Analyzing a 7 dimensional plane simplex with  $m = 7$ , we can start with its 3 dimensional subsimplices. We see that they contain points corresponding to partitions: 7,0,0; 6,1,0; 5,2,0; 4,3,0; 5,1,1; 4,2,1; 3,3,1; 3,2,2. The points corresponding to partitions are connected with other points of the simplex by circles. In higher dimensions the circles become spheres and this is the reason why we call a partition an *orbit*. The other points on each orbit have only different ordering of the same set of the coordinates.

Arranging partitions into tables (Table 11), the column classification is made according to the number of nonzero parts of partitions. Another classifying criterion is needed for rows. This will be the length of the longest vector  $m_1$ . From all partition vectors having the same dimensionality the longest vector is that one with the longest first vector. It dominates them. But there can exist longer orbits nearer to the surface of the simplex with a lesser number of nonzero parts. For example, vector (4,1,1) has equal length as (3,3,0) but

Table 11: Partition scheme (7,7)

n	1	2	3	4	5	6	7	$\Sigma$
m = 7	1							1
6		1						1
5		1	1					2
4		1	1	1				3
3			2	1	1			4
2				1	1	1		3
1							1	1
$\Sigma$	1	3	4	3	2	1	1	11

vector (4,1,1,1,1) is shorter than (3,3,2,0,0). Such an arrangement is on Table 11. Orbits with three nonzero parts lie inside the 3 dimensional simplex, with two nonzero parts lie on its edges. Orbits with four nonzero parts are inside tetrahedrons, it is on a surface in the fourth dimension. There exist these partitions: 4,1,1,1; 3,2,1,1; 2,2,2,1. Similarly columns corresponding to higher dimensions are filled.

The rows of partition schemes classify partitions according to the length of the first and longest vector  $\mathbf{e}_1$ . It can be shown easily that all vectors in higher rows are longer than vectors in lower rows in corresponding columns. In the worst case it is given by the difference

$$(x+1)^2 + (x-1)^2 > (2x)^2. \quad (31)$$

A three dimensional plane simplex to be can be considered as a truncated 7 dimensional simplex, and after completing the columns of the Tab. 11) by the corresponding partitions, we get a crosssection through the 7 dimensional plane. The analysis is not perfect, an element is formed by two orbits, but nevertheless the scheme gives an insight how such high dimensional space looks like. We will study therefore properties of partitions schemes thoroughly.

The number of nonzero vectors in partitions will be given as  $n$ , the size of the first vector as  $m$ . Zeroes will not be written to spare work. The bracket  $(m, n)$  means all partitions of the number  $m$  into at most  $n$  parts. Because we

Table 12: Partition scheme  $m = 13$ 

n	1	2	3	4	5	6	7	8	9	10	11	12	13
m=13	1												
12		1											
11		1	1										
10		1	1	1									
9		1	2	1	1								
8		1	2	2	1	1							
7		1	3	3	2	1	1						
6			3	4	3	2	1	1					
5			2	4	5	3	2	1	1				
4				3	4	4	3	2	1	1			
3					2	3	3	2	2	1	1		
2							1	1	1	1	1	1	
1													1
$\Sigma$	1	6	14	18	18	14	11	7	5	3	2	1	1

write a partition as a vector, we allow zero parts to complete the partition as before.

### 13 Construction of Partition Schemes

A partition scheme is divided into four blocks. Diagonal blocks repeat the Table 4.1 (the left upper block), the right lower one is written in the transposed form for  $n > m/2$ . Odd and even schemes behave somewhat differently, as can be seen on Tables 12 and 13.

In the left lower block nonzero elements indicated by asterisks \* can be placed only over the line which gives sufficient great product  $mn$  to place all units into the corresponding Ferrers graphs and their sums must agree not only with row and column sums, but with diagonal sums, as we show below. This can be used for calculations of their numbers, together with rules for restricted partitions.

The examples show three important properties of partition schemes:

- Partition schemes are symmetrical according to their transversals, due to

Table 13: Partition scheme  $m = 14$ 

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
m=14	1													
13		1												
12		1	1											
11		1	1	1										
10		1	2	1	1									
9		1	2	2	1	1								
8		1	3	3	2	1	1							
7		1	3	4	3	2	1	1						
6			3	*	*	*	2	1	1					
5			1	*	*	*	3	2	1	1				
4				3	*	*	4	3	2	1	1			
3					2	*	3	3	2	2	1	1		
2							1	1	1	1	1	1	1	
1														1
$\Sigma$	1	7	16	23	23	20	15	11	7	5	3	2	1	1

the conjugated partitions obtained by transposing Ferrers graphs.

- The upper left quarter (transposed lower right quarter) contain elements of the Table 4.1 of partitions into exactly  $n$  parts shifted one column up.
- The schemes have form of the matrix in the lower diagonal form with unit diagonal. Therefore, they have inverses. It is easy to find them, for example for  $n = 7$  (Table 14).

The partitions in rows must be balanced by other ones with elements of inverse columns. The third column includes or excludes 331 and 322 with 3211 and  $31^4$ ;  $2^31$  and  $2^21^3$  with  $2 \times 21^5$ , respectively.

## 14 Lattices of Orbits

Partition orbit is a sphere which radius  $r$  is determined by the Euclidean length of the corresponding vector:  $r = (\sum p_j^2)^{1/2}$ . Radiuses of some partition orbits coincide, for example  $r(3, 3, 0)^2 = r(4, 1, 1)^2 = (18)$ . It is thus impossible to

Table 14: Partition scheme (7,7) and its inversion

n	1	2	3	4	5	6	7	1	2	3	4	5	6	7
m = 7	1							1						
6		1							1					
5		1	1						0	1				
4		1	1	1					0	-1	1			
3			2	1	1				2	-1	-1	1		
2				1	1	1			-2	2	0	-1	1	
1							1		0	0	0	0	0	1

determine distances between orbits using these radii (Euclidean distances) since the distance between two different orbits cannot be zero.

We have shown in Sect. 5 that one orbit can be obtained from another by shifting just two vectors, one up and other down on the number scale. We can imagine that both vectors collide and exchange their values as two particles of the ideal gas exchange their energy. If we limit the result of such an exchange to 1 unit, we can consider such two orbits to be the nearest neighbor orbits. The distance inside this pair is  $\sqrt{2}$ . We connect them in the scheme by a line. Some orbits are thus connected with many neighbor orbits, other have just one neighbor, compare with Fig. 5. Orbits (3,3,0) and (4,1,1) are not nearest neighbors, because they must be transformed in two steps:

$$(3, 3, 0) \leftrightarrow ((3, 2, 1) \leftrightarrow (4, 1, 1))$$

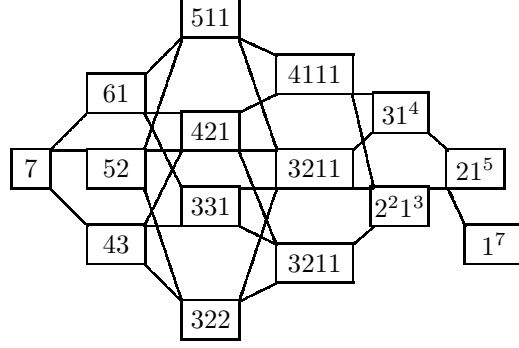
or

$$(3, 3, 0) \leftrightarrow (4, 2, 0) \leftrightarrow (4, 1, 1) .$$

Partition schemes are generally not suitable for construction of orbit lattices, because at  $m = n > 7$  there appear several orbits on some table places. It is necessary to construct at least 3 dimensional lattices to show all existing connections. For example:



Figure 6: Lattice of file partitions. A file can be split into two new ones or two files can be combined into one



$$\begin{array}{ccccc}
 (5, 2, 1) & \leftrightarrow & (4, 3, 1) & \leftrightarrow & (3, 3, 2) \\
 & \searrow \swarrow & \updownarrow & \swarrow \nwarrow & \\
 & & (4, 2, 2) & & 
 \end{array}$$

Sometimes stronger condition are given on processes going at exchanges, namely, that each collision must change the number of empty parts, as if they were information files which can be only joined into one file or one file separated into two or more files, or as if a part of a file transferred into an empty file. Here also the nearest neighbor is limited on unifying of just 2 files or splitting a file into two (Fig.6). In this case the path between two orbits must be longer, for example:

$$(3, 3, 0) \leftrightarrow (6, 0, 0) \leftrightarrow (4, 2, 0) \leftrightarrow (4, 1, 1)$$

or

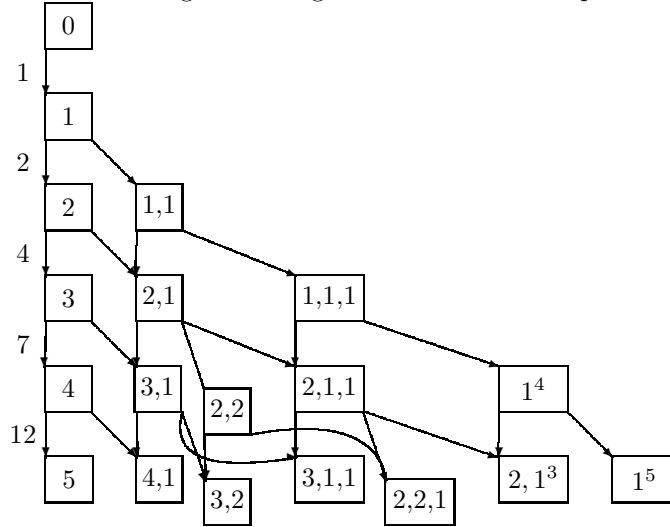
$$(3, 3, 0) \leftrightarrow (3, 2, 1) \leftrightarrow (5, 1, 0) \leftrightarrow (4, 1, 1) .$$

In a lattice it is possible to count the number of nearest neighbors. If we investigate the number of one unit neighbors or connecting lines between columns

Table 15: Right hand One-unit Neighbors of Partition Orbits

n	1	2	3	4	5	6	$\Sigma$
m=2	1						1
3	1	1					2
4	1	2	1				4
5	1	3	2	1			7
6	1	4	4	2	1		12
7	1	5	6	4	2	1	19
D(7-6)	0	1	2	2	1	1	7

Figure 7: Neighbor lattices between plane simplexes



of partition schemes, we obtain an interesting Table 15.

The number of right hand neighbors is the sum of two terms. The right hand diagonal neighbors exist for all  $p(m, n - 1)$ . We add 1 to all these partitions and decrease the largest part. Undetermined remain right hand neighbors in rows. Their number is equal to the number of partitions  $p(m - 2)$ . To each partition  $p(m - 2, n - 1)$  are added two units, one in the  $n$  th column, the second in the  $(n - 1)$  the column.

The number of right hand neighbors  $P(n)$  is the sum of the number of un-

restricted partitions

$$P(n) = \sum_{k=0}^{n-2} p(k) . \quad (32)$$

To find all neighbors, we must add neighbors inside columns. The number of elements in columns is the number of partitions into exactly  $n$  parts  $p(m, n)$ , the difference of each column must be decreased by 1 but there exist additional connections, see Fig. 7.

These connections must be counted separately. The resulting numbers are already known. The construction of partition schemes gives the result which we know as Table 1 read from the diagonal to the left.

The other interpretation of right hand one-unit neighbors of partitions is the plane complex as on Fig. 7. Vectors connect nearest neighbors in layers.

## 15 Diagonal Differences in Lattices

In lattices, we can count orbits on side diagonals going consecutively parallel to the main diagonal. They count orbits having the form  $[n - k]^k$ . Their Ferrers graphs have a L form

$$\begin{array}{cccc} x & x & x & x \\ x & & & \\ x & & & \\ x & & & \end{array}$$

Side diagonal elements counts partitions which have in this layer smaller number of units, the other are inside this base.

The corresponding Table is 16.

The initial  $k$  column values have these analytical forms:

- $1n$  counts elements in  $n$  columns (rows) having the form  $(n - k)1^k$ ,  $k = 0 - (n - 1)$ ;
- $1(n-3)$  counts elements in  $(n - 2)$  columns (rows) obtained from the basic partition 2,2 by adding units in the first row and column;

Table 16: Diagonal Sums of Partitions

k	1	2	3	4	5	6	7	8	9	$\Sigma$
n= 1	1									1
2	2									2
3	3									3
4	4	1								5
5	5	2								7
6	6	3	2							11
7	7	4	4							15
8	8	5	6	3						22
9	9	6	8	6	1					30
10	10	7	10	9	6					42
11	11	8	12	12	11	2				56
12	12	9	14	15	16	9	2			77
13	13	10	16	18	21	16	7			101
14	14	11	18	21	26	23	18	4		135
15	15	12	20	24	31	30	29	12	3	176

- $2(n-5)$  counts elements in  $(n-2)$  columns (rows) obtained from the basic partitions 3,3 and 2,2,2 by adding units in the first row and column;
- $3(n-7)$  counts elements in  $(n-2)$  columns (rows) obtained from the basic partitions 4,4; 3,3,2, and 2,2,2,2 by adding units in the first row and column;
- $5(n-9) + 1$ . On this level appears the partition 3,3,3 where elements start to occupy the third L layer;
- $7(n-11) + 2$ .

The values in brackets are the numbers for partitions which lie inside the L frame having  $(2k-1)$  units. At higher diagonal layers appear these possibilities to add new elements later. Partitions 4, 4, 4 and 3, 3, 3, 3, for  $n = 12$ , are counted in the seventh layer. For  $n = 13$ , the layer counts seven partitions:

5, 5, 3;  
 5, 4, 4;  
     4, 4, 4, 1;  
     4, 4, 3, 2;  
     4, 3, 3, 3;  
         3, 3, 3, 3, 1;  
         3, 3, 3, 2, 1.

There appears a very interesting property of partition lattices. The side diagonals being on side diagonals of the Table 16 have equal length  $n$ , and the number of partitions  $p(d)$  lying on them is equal to

$$p(d) = 2^{(n-1)} \quad (33)$$

this is true for all complete diagonals in the table, also the seventh diagonal sum is completed by the partition (4,4,4,4). It can be conjectured, that it is a general property of lattices. There are counted partitions which superposed Ferrers graphs can be situated into isoscele triangular form ( $M = N$ ) ending in the transversal which were not counted before. The condition is that all Ferrers graphs are superposed from the same starting place, otherwise Ferrers graphs of each partition can fill their isoscele triangular form.

The partitions can be ordered in the following way (see Table 17)

Counting of partitions is changed into a combinatorial problem of finding of all ordered combinations of  $(k - 1)$  numbers with the greatest part equal to  $k$ . The partitions are formed by a fixed part which is equal to the number of the column and starts in the corresponding row. To this fixed part are added two movable parts from the previous partitions, the whole upper predecessor and the movable part of the left upper predecessor. The resulting counts are the binomial numbers.

## 16 Generalized Lattices

Table 17: Binomial Ordering of Partitions

1	2	3	4	5	$\Sigma$
(1)					1
(1,1)	(2)				2
(1,1,1)	(2,1) (2,2)	(3)			4
(1,1,1,1)	(2,1,1) (2,2,1) (2,2,2)	(3,1) (3,2) (3,3)	(4)		8
(1,1,1,1,1)	(2,1,1,1) (2,2,1,1) (2,2,2,1) (2,2,2,2)	(3,1,1) (3,2,1) (3,2,2) (3,3,1) (3,3,2) (3,3,3)	(4,1) (4,2) (4,3) (4,4)	(5)	16

Figure 8: Nearest neighbors in 00111 lattice

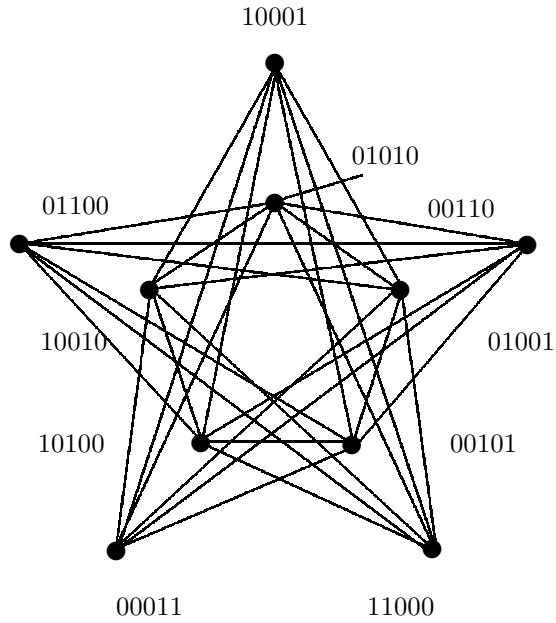
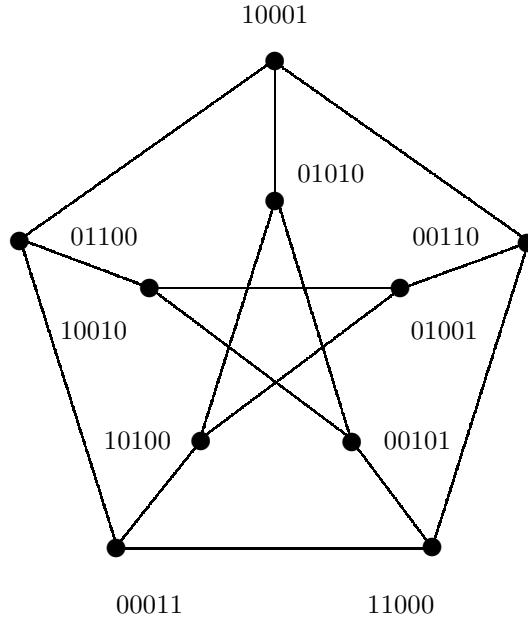


Figure 9: Petersen graph. Adjacent vertices are in distances 4



The notion of lattices can be used also for possible transformations of points having specific properties among themselves, for example between all 10 permutations of a 5 tuple composed from 3 symbols of one kind and 2 symbols of another kind. When the neighbors differ only by one exchange of the position of only anyone pair of two kinds symbols we obtain lattice as on Fig.8. Each from three unit symbols has two possibilities to change 0 into 1. Rearrange these ten points as a simple triangle. The simultaneous exchange of two pairs (or two consecutive changes of one pair give a pattern as on Fig.9, known as the Petersen graph.

Lattices are formed by vertices of  $n$  dimensional cubes. The nearest vertices differ only by one coordinate. The lattices of the 3 dimensional cube is on Fig. 10. Compare lines of the graphs with a real 3 dimensional cube and try to

Figure 10: Lattice of the three dimensional unit cube

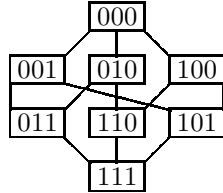
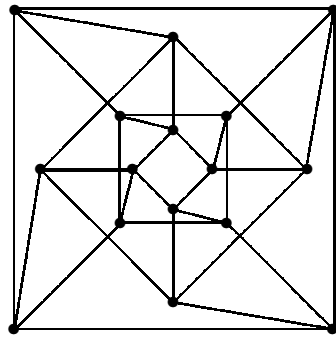


Figure 11: Four dimensional cube projection. One 3 dimensional cube is twisted  $45^\circ$



imagine the 4 dimensional cube (Fig. 11).

A classical example of relation lattices is Aristotle's attribution of four properties: **warm**, **cold**, **dry**, and **humid** to four elements: fire, air, water and earth, respectively. It can be arranged in a form

<i>air</i>	<b>humid</b>	<i>water</i>
<b>warm</b>	0	<b>cold</b>
<i>fire</i>	<b>dry</b>	<i>earth</i> .

The elements have always only two properties. The properties adjacent vertically and horizontally exclude themselves. Something can not be simultaneously



warm and cold, or humid and dry<sup>2</sup>.

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<sup>2</sup>More precisely, it is necessary to draw a borderline (point zero) between these properties. Depending on its saturation, water vapor can be dry as well as wet.